Unified model for network dynamics exhibiting nonextensive statistics

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We introduce a dynamical network model which unifies a number of network families which are individually known to exhibit q-exponential degree distributions. The present model dynamics incorporates static (nongrowing) self-organizing networks, preferentially growing networks, and (preferentially) rewiring networks. Further, it exhibits a natural random graph limit. The proposed model generalizes network dynamics to rewiring and growth modes which depend on internal topology as well as on a metric imposed by the space they are embedded in. In all of the networks emerging from the presented model we find q-exponential degree distributions over a large parameter space. We comment on the parameter dependence of the corresponding entropic index q for the degree distributions, and on the behavior of the clustering coefficients and neighboring connectivity distributions.

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I. INTRODUCTION

Over the past two decades, nonextensive statistical mechanics has successfully addressed a wide spectrum of nonequilibrium phenomena in nonergodic and other complex systems [1,2]. Recently, it has also entered the field of complex networks [3–10]. Nonextensive statistical mechanics is a generalization of Boltzmann-Gibbs (BG) statistical mechanics. It is based on the entropy

$$S_q \equiv \frac{1-\int dx[p(x)]^q}{q-1} \quad \left(S_1 = S_{\rm BG} \equiv -\int dx p(x) \ln p(x)\right). \tag{1}$$

The extremization of the entropy S_q under appropriate constraints [11] yields the stationary-state distribution. This is of the q-exponential form, where the q-exponential function is defined as

$$e_q^x \equiv [1+(1-q)x]^{1/(1-q)}, \tag{2}$$

for $1+(1-q)x \geq 0$, and zero otherwise (with $e_1^x = e^x$). The tail exponent $\gamma \equiv 1/(q-1)$ characterizes the asymptotic power-law distribution.

Since the very beginning of the tremendous recent modeling efforts of complex networks it has been noticed that degree distributions asymptotically follow power laws [12], or even exact q exponentials [13]. The model in [12] describes growing networks with a so-called preferential attachment rule, meaning that any new node i being added to the system links itself to an already existing node j in the network with a probability that is proportional to the degree k_j of node j. In [13] this model was extended to also allow for preferential rewiring. The analytical solution to the model has an exact q-exponential result, $P(k) \propto \left[1-(1-q)\frac{k}{l}\right]^{1/1-q}$, where the nonextensivity parameter q and the power onset l are defined in terms of the parameters κ and γ used in [13] by $q \equiv 1+1/\gamma$ and $l \equiv (1-q)\kappa$, for details see the last paragraph of Sec. III.

Many real world networks, like the internet, air travel, railway, etc., are embedded in metric spaces. This has recently been accounted for in the literature to some extent, see, e.g., [14–16]. Quite similar to [14] in [3] preferential attachment networks have been embedded in Euclidean space, where the attachment probability for a newly added node is not only proportional to the degrees of existing nodes, but also depends on the Euclidean distance between nodes. The model is realized by setting the linking probability of a new node to an existing node i to be $p_{\rm link} \propto k_i / r_i^{\alpha}$ ($\alpha \geq 0$), where r_i is the distance between the new node and node i; $\alpha = 0$ corresponds to the model in [12] which has no metrics. The analysis of the degree distributions of the resulting networks has exhibited [3] q exponentials with a clear α dependence of the nonextensivity parameter q. In the large α limit, q approaches unity, i.e., random networks are recovered in the Boltzmann-Gibbs limit. In an effort to understand the evolution of socioeconomic networks, a model was proposed in [6] that builds upon [13] but introduces a rewiring scheme which depends on the *internal* network distance between two nodes, i.e., the number of steps needed to connect the two nodes. The emerging degree distributions have been subjected to a statistical analysis where the (null) hypothesis of q exponentials could not be rejected.

It has been found that networks exhibiting degree distributions compatible with q exponentials are not at all limited to growing and preferentially organizing networks. A model for nongrowing networks which was recently put forward in [4] also unambiguously exhibits q-exponential degree distributions. This model was motivated by interpreting networks as a certain type of gas where upon an (inelastic) collision of two nodes, links get transferred in analogy to the energy-momentum transfer in real gases. In this model a fixed num-

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ber of nodes in a (undirected) network can merge, i.e., two nodes fuse into one single node, which keeps the union of links of the two original nodes; the link connecting the two nodes before the merger is removed. At the same time a new node is introduced to the system and is linked randomly to any of the existing nodes in the network [17]. Due to the nature of this model the number of links is not strictly conserved—which can be thought of as jumps between discrete states in some phase space. The model has been further generalized to exhibit a distance dependence as in [3], however, r_i is not the Euclidean but is rather the internal distance. Again, the resulting degree distributions have q-exponential form [4].

A quite different approach was taken in [18] where an ensemble interpretation of random networks has been adopted, motivated by superstatistics [19]. Here it was assumed that the average connectivity $\bar{k}$ in random networks is fluctuating according to a distribution $\Pi(\bar{k})$, which is sometimes associated with a hidden-variable distribution. In this sense a network with any degree distribution can be seen as a superposition of random networks with the degree distribution given by $p(k)=\int_0^\infty d\lambda \Pi(\bar{k})\frac{\bar{k}^k e^{-\bar{k}}}{k!}$. It was shown in [18], that an asymptotically power-law functional form of the hidden variable distribution, $\Pi(\bar{k}) \propto \bar{k}^{-\gamma}$, leads to a q-exponential degree distribution. This is an exact result. More recently a possible connection between *small-world* networks and the maximum S_q-entropy principle, as well as to the hidden variable method [18], has been noticed in [9].

In yet another view, networks have recently been treated as statistical systems on a Hamiltonian basis [21–24]. It has been shown that these systems show a phase-transition-like behavior [22], along which network structure changes. In the low temperature phase one finds networks of star type, meaning that a few nodes are extremely well connected resulting even in a discontinuous $p(k)$; in the high temperature phase one finds random networks. Surprisingly, for a special type of Hamiltonian, networks with q-exponential degree distributions emerge right at the transition point [24].

Given the above characteristics of networks and the fact that a vast number of real-world and model networks show asymptotic power-law degree distributions, it seems quite natural to look for a deeper connection between networks and nonextensive statistical physics. The mere fact that networks are intrinsically nonextensive can already be seen in a simple combinatorial argument, given, e.g., in [25]. The purpose of this work is to show that various model types can be unified into a single dynamic network-formation model, characterized by a reasonably small number of parameters. We show explicitly how a number of famous network formation models appear as special cases of the proposed model, ranging from random graphs [27] to highly nontrivial networks, showing not only power-law degree distributions but also nontrivial clustering and neighboring connectivity. Maybe the most remarkable finding is that, within the parameter space of the proposed model, all these networks types seem to be compatible with q-exponential degree distributions. A clear example is the model introduced in [13]. Indeed, the authors analytically obtain $p(k) \propto (k+k_0)^{-\gamma}$, which can be rewritten in the q-exponential form as $p(k) \propto e_q^{-k/l}$ with $q=\frac{\gamma+1}{\gamma}$ and $l=k_0(q-1)$. Other network models follow the same path, although the results are only numerical. The emergence, for networks, of the basic distribution within nonextensive statistical mechanics, is not so surprising after all. Indeed, if we associate to each link an "energy" (or *cost*) and to each node one-half of the "energy" carried by its links (the other one-half being associated with the other nodes to which any specific node is linked), the distribution of energies optimizing S_q precisely coincides with the degree distribution. If, for any reason, we consider k as the modulus of a d-dimensional vector $\mathbf{k}$, the optimization of the functional $S_q[p(\mathbf{k})]$ may lead to $p(k) \propto k^\eta e_q^{-k/l}$, where k^η plays the role of a density of states, $\eta(d)$ being either zero, or positive or negative.

II. MODEL

The following model is a unification and generalization of the models presented in [3,4]. The model in [3] captures preferential growing aspects of networks embedded into a *metric space*, while [4] introduces a static, self-organizing model with a sensitivity to an *internal metric* (chemical distance, Diekstra distance). The rewiring scheme there can be thought of as having preferential attachment aspects in one of its limits [17] (see below), but has none in the other limit.

The network evolves in time as described in [3]: At $t=1$, the first node $(i=1)$ is placed at some arbitrary position in a metric space. The next node is placed isotropically on a sphere (in that space) of radius r, which is drawn from a distribution $P_G(r) \propto 1/r^{\alpha_G}$ $(\alpha_G > 0)$, G stands for *growth*. To avoid problems with the singularity, we impose a cutoff at $r_{\min}=1$. The second node is linked to the first. The third node is placed again isotropically on a sphere with random radius $r \in P_G$, however, the center of the sphere is now the barycenter of all the preexisting nodes. From the third added node on, there is an ambiguity where the newly positioned node should link to. We choose a generalized preferential attachment process, meaning that the probability that the newly created node i attaches to a previously existing node j is proportional to the degree k_j of the existing node j, and on the metric (Euclidean) distance between i and j, denoted by r_{ij}. In particular the linking probability is

$$p_{ij}^A = \frac{k_j/r_{ij}^{\alpha_A}}{\sum_{j=1}^{N(t)-1} k_j/r_{ij}^{\alpha_A}}, \tag{3}$$

where $N(t)$ is the number of nodes at time t. It is not necessary that at each time step only one node is entering the system, so we immediately generalize that a number of $\bar{n}$ nodes are produced and linked to the existing network with $\bar{l}$ links per time step. Note that $\bar{n}$ and $\bar{l}$ can also be random numbers from an arbitrary distribution. For simplicity and clarity we fix $\bar{n}=1$ and $\bar{l}=1$.

After every λ time step, a different action takes place on the network. At this time step the network does not grow but a pair of nodes, say i and j, merge to form one single node

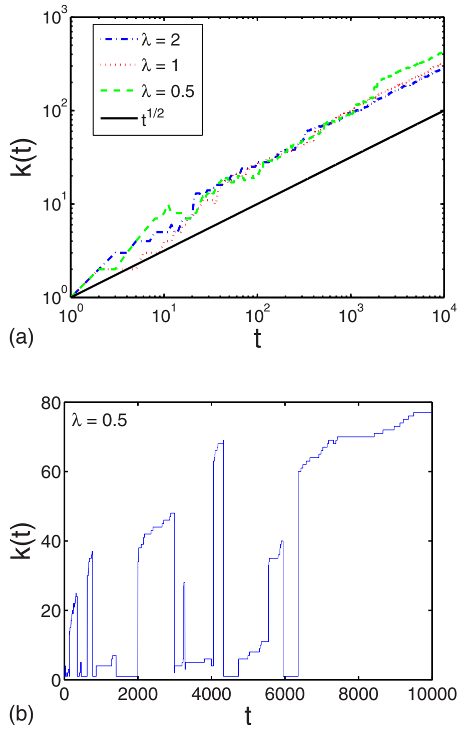

FIG. 1. (Color online) (a) Time evolution of the degree of the best-connected node for the parameters, $N=10000$, $\alpha_A=0$, $\alpha_M=0$ and three values of λ. Asymptotically $k(t)$ approaches the mean-field result [28], i.e., a square root behavior (straight line); the superimposed jumps are due to the merging processes. (b) Same as the randomly chosen node. Large downward jumps in the degree occur when the node merges and loses all of its links to the other node. Large upward jumps are due to a merging process where the other node loses all of its links to the randomly chosen node under investigation. Small jumps mean that either the node merges with small degree nodes or that it receives new links via the attachment mechanism.

[17]. This node keeps the name of one of the original nodes, say, for example, i. This node now gains all the links of the other node j, resulting in a change of degree for node i according to

$$k_i \rightarrow k_i + k_j - N_{\mathrm{common}} \quad \text{if } (i,j) \text{ are not first neighbors},$$

$$k_i \rightarrow k_i + k_j - N_{\mathrm{common}} - 2 \quad \text{if } (i,j) \text{ are first neighbors}, \tag{4}$$

where N_{common} is the number of nodes, which shared links to both of i and j before the merger. In the case that i and j were first neighbors before the merger, i.e., they had been previously linked, the removal of this link will be taken care of by the term −2 in Eq. (4). The probability that two nodes i and j merge can be made distance dependent, as before. In particular, to stay close to the model in [4], we choose the following procedure. We randomly choose node i with probability $\propto 1/N(t)$ and then choose the merging partner j with probability

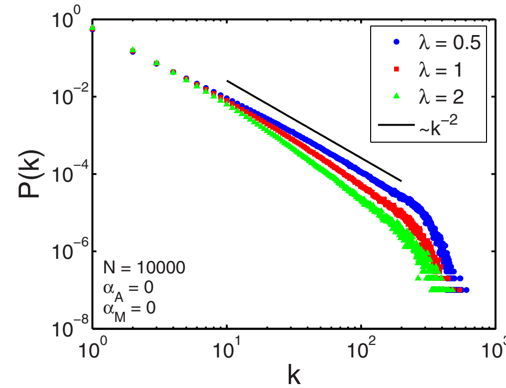

FIG. 2. (Color online) Degree distribution $P(k)$ (unnormalized) for $N=10\,000$, $\alpha_A=0$, $\alpha_M=0$ and various values of λ. This case corresponds to a growing network with preferential linking and random merging of nodes. We use it to show the effect of λ on the decay exponents. For larger values of λ the exponents become larger. The case $\alpha_A \neq 0$, $\alpha_M \neq 0$ computationally limits the size of the networks.

$$p_{ij}^{M} = \frac{d_{ij}^{-\alpha_M}}{\sum_j d_{ij}^{-\alpha_M}} \quad (\alpha_M \geq 0), \tag{5}$$

where d_{ij} is the shortest distance (path) on the network connecting nodes i and j; Obviously, tuning α_M from 0 toward large values, switches the model from the case where j is picked fully at random $[\propto 1/N(t)]$, to a case where only nearest neighbors of i will have a non-negligible chance to get chosen for the merger. Note that the number of nodes is reduced by one at that point. To keep the number of nodes constant at this time step, a new node is introduced and linked with $\bar{l}$ of the existing nodes with probability given in Eq. (3).

The relevant model parameters are the merging exponent α_M, the attachment exponent α_A, controlling the sensitivity of distance in the network, and the relative rate of merging and growing, λ. We explicitly checked that the remaining parameters, α_G, $\bar{n}$, $\bar{l}$, and r_{min} play marginal roles in the dynamics of the model. In particular, the distribution of points in space as governed by α_G does practically not influence the resulting degree distributions, in the range of $\alpha_G \in [1,3]$.

We simulate this model and record the degrees k_i, the clustering coefficients c_i (defined below), and the nearest-neighbor connectivity k_i^{nn}, for all individual nodes i (Fig. 1). From these values we derive distribution functions (as a function of k). In Fig. 2 typical degree distributions are shown for three typical values of λ. Obviously, the distribution is dominated by a power-law decay (see details of the functional form below) ending in an exponential finite size cutoff for large k.

The clustering coefficient of node i, c_i is defined by

$$c_i = \frac{2e_i}{k_i(k_i-1)}, \tag{6}$$

with e_i being the number of triangles node i is part of. $c(k)$ is obtained by averaging over all c_i with a fixed k. It has been noted that $c(k)$ contains information about hierarchies present in networks [26]. For Erdös-Rényi (ER) networks [27], as well as for pure preferential attachment algorithms without the possibility of rewiring, the clustering coefficient $c(k)$ vs degree is flat. The global clustering coefficient is the average over all nodes $C=\langle c_i \rangle_i$. A large global clustering coefficient is often used for identification of the small-world structure [20]. The average nearest-neighbor connectivity (of the neighbors) of node i is

$$k_i^{\mathrm{nn}} = \frac{1}{k_i} \sum_{j \text{ neighbor of } i} k_j. \tag{7}$$

When plotted as a function of k, $k^{\mathrm{nn}}(k)$ is a measure to assess the assortativity of networks. A rising function means assortativity, which is the tendency for well-connected nodes to link to other well-connected ones, while a declining function signals disassortative structure. Depending on the variables of the model, known networks result as natural limits.

(1) Soares and co-workers limit. For the lim $\lambda \to \infty$ we have no merging, and α_M is an irrelevant parameter. The model corresponding to this limit has been proposed and studied in [3].

(2) Albert-Barabasi limit. The lim $\lambda \to \infty$ and lim $\alpha_A \to 0$, gets rid of the metric in the Soares and co-workers model and recovers the original Albert-Barabasi preferential attachment model [12].

(3) Kim and co-workers limits. The limit lim $\lambda \to 0$ allows no preferential growing of the network. If at each time step after every merger a new node is linked randomly with $\bar{l}$ links to the network, the model reported in [4] is recovered. The lim $\lambda \to 0$ model with lim $\alpha_M \to 0$ (lim $\alpha_M \to \infty$) recovers the *random* case (*neighbor* case) in [17].

III. NONEXTENSIVE CHARACTERIZATION OF COMPLEX NETWORKS

There has been a convincing body of evidence that for a large class of networks (normalized), degree distributions can be fit by q exponentials,

$$P(k) = e_q^{-(k-1)/\kappa} \quad (k = 1,2,3,4,\ldots), \tag{8}$$

where the q-exponential function is defined in Eq. (2), with $q \geq 1$, and $\kappa > 0$ some characteristic number of links. A convenient procedure to perform a two-parameter fit of this kind is to take the q logarithm of the distribution P, defined by $Z_q(k) \equiv \ln_q P(k) \equiv \frac{[P(k)]^{1-q}-1}{1-q}$. This is done for a series of different values of q. The function $Z_q(k)$ which can be best fit (least squares) with a straight line determines the value of q, the slope being $-\kappa$. Note, that a least-squares fit of $Z_q(k)$ corresponds to logarithmically weighted errors in $P(k)$.

In Fig. 3 we show the degree distribution for several system sizes together with the q logarithm $Z_q(k)$, from which an optimum q and κ can be obtained. We conclude that, with good precision, the *Ansatz* in Eq. (8) for the degree distribution, when seen as a null hypothesis, cannot be rejected on the basis of a χ^2 statistics for any reasonable significance level, for the system sizes studied.

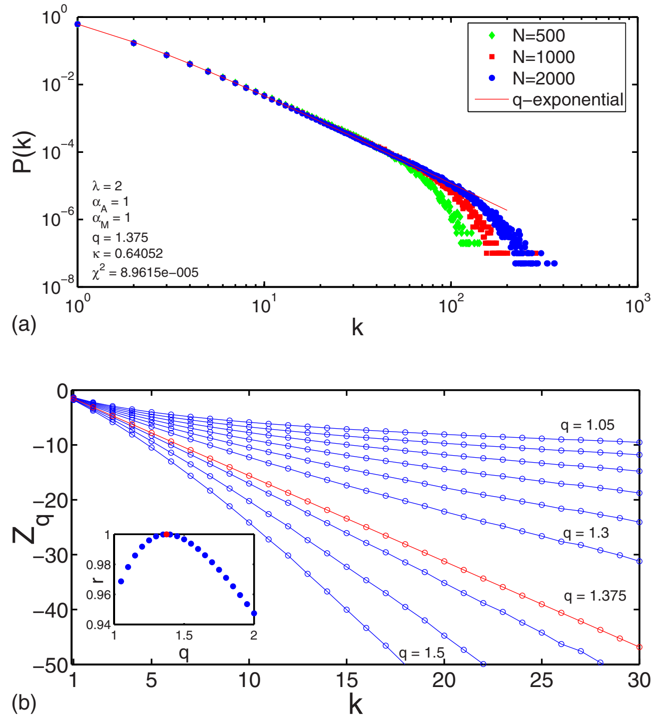

FIG. 3. (Color online) (a) $P(k)$ for $\lambda=2$, $\alpha_A=1$, $\alpha_M=1$, and various system sizes (symbols). The line is the q-exponential fit for $N=2000$. (b) q logarithm of the (normalized) $P(k)$ from (a). The line associated with $q=1.375$ corresponds to an optimal linear fit, i.e., a maximum of the correlation coefficient (inset) of a straight line with Z_q. The quality of the fit in (a) is given by standard χ^2 statistics.

For actual curve fitting, it is often more convenient to use the *cumulative* distributions, which can be parametrized by

$$P(\geq k) = e_{q_c}^{-(k-1)/\kappa_c} \quad (k = 1,2,3,4,\ldots). \tag{9}$$

On the other hand, the corresponding cumulative distribution $P(\geq k)$ is given by (we switch to integral notation for simplicity for a moment)

$$P(\geq k) \equiv 1 - \int_1^k dk' P(k') = \left(1 - \frac{1-q}{\kappa}(k-1)\right)^{2-q/1-q}. \tag{10}$$

By comparison of coefficients the cumulative parameters are given by

$$q_c = \frac{1}{2-q} \quad \text{and} \quad \kappa_c = \frac{\kappa}{2-q}. \tag{11}$$

Whenever we talk about q values corresponding to a cumulative distribution, we use the notation q_c and κ_c, where c indicates *cumulative*.

In addition to the q-exponential fits we have fitted the degree distributions with pure powers and pure exponential (fixed $q=1$), and obtained much worse values for χ^2. The remarkable quality of q-exponential fits to the degree distributions from the model, reveals a connection [3] of scale-free network dynamics to nonextensive statistical mechanics

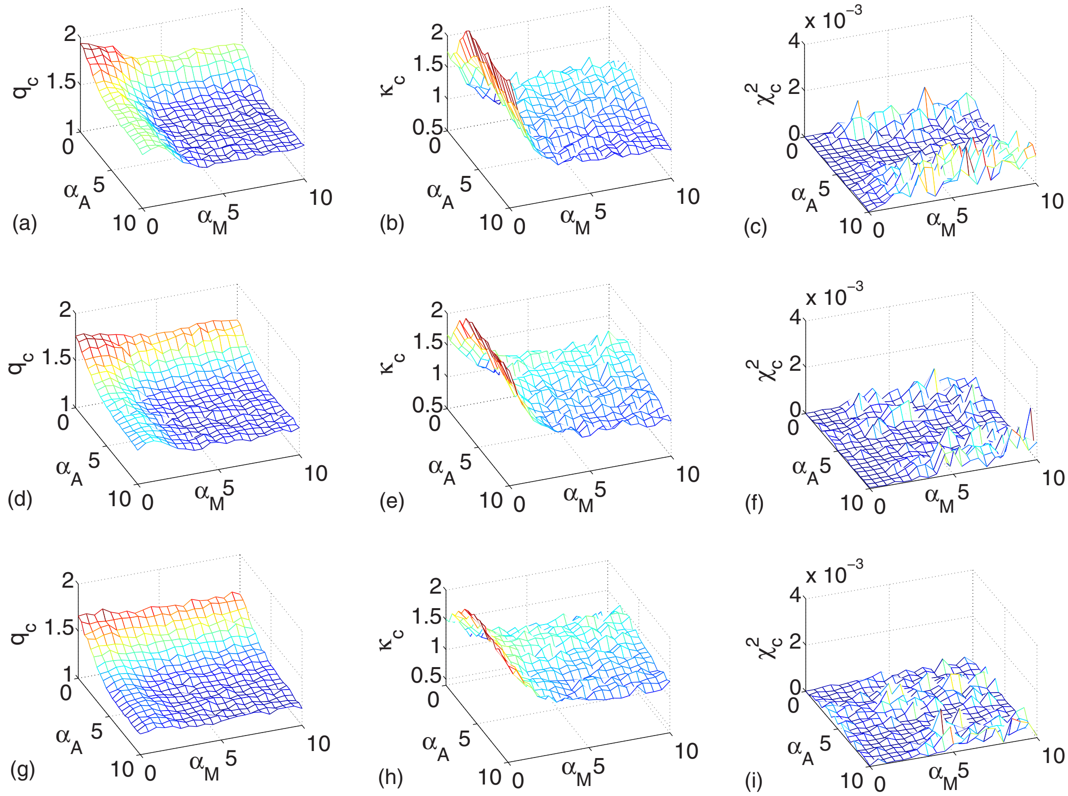

FIG. 4. (Color online) q_c (left-hand column) and κ_c (center column) values from q-exponential fits to the cumulative degree distributions $P(>k)$ for $\alpha_G=1$, $N=1000$, and $\lambda=0.5$ (top), $\lambda=1$ (middle), $\lambda=2$ (bottom). The fit quality is given by the χ^2 value per degree of freedom (right-hand column).

[1,2]. To make the point more clear, consider the entropy

$$S_q \equiv \frac{1-\int_1^\infty dk[p(k)]^q}{q-1} \quad \left(S_1 = S_{\mathrm{BG}} \equiv -\int_1^\infty dk p(k) \ln p(k)\right), \tag{12}$$

where we assume k as a continuous variable for simplicity. If we extremize S_q with the constraints [11]

$$\int_1^\infty dk p(k) = 1 \quad \text{and} \quad \frac{\int_1^\infty dk k[p(k)]^q}{\int_1^\infty dk[p(k)]^q} = K, \tag{13}$$

we obtain

$$p(k) = \frac{e_q^{-\beta(k-1)}}{\int_1^\infty dk' e_q^{-\beta(k'-1)}} = \beta(2-q)e_q^{-\beta(k-1)} \quad (k \geq 1), \tag{14}$$

where β is determined through Eq. (13). Both positivity of $p(k)$ and the normalization constraint (13) impose $q<2$.

Let us emphasize again that models do exist that can be handled analytically, and which exhibit precisely q-exponential degree distributions. Such is the case of [13], where the degree distribution is of the form $p(k) \propto (k+k_0)^{-\gamma}$, which can be rewritten as a q exponential with $q=\frac{\gamma+1}{\gamma}=\frac{2m(2-r)+1-p-r}{m(3-2r)+1-p-r}$, where (m,p,r) are parameters of the particular model. It is important to stress that the model of [13] is *not* a special case of our model, even in the case $\alpha_A=\alpha_M=0$ (preferential attachment and random merging) the two models cannot directly be mapped into each other. It is however interesting that both fall into the same universality class of distribution functions.

IV. RESULTS

Realizing the above network model in numerical simulations we compute degree distributions, clustering coefficients, and neighbor connectivity, for a scan over the relevant parameter space, spanned by λ, α_A, and α_M. All of the following data were obtained from averages over 100 identical network realizations with a final $N(t^{\max})=1000$; for finite size checks we have included runs with $N(t^{\max})=500$ and 2000. The reason for these relatively modest network sizes is that, at every time step, all network distances must be evaluated. The remaining parameters have been checked to be of marginal importance and have been fixed to $\alpha_G=1$, $\bar{n}=1$, and $\bar{l}=1$.

The fitted values for the nonextensive index q_c and the characteristic degree κ_c are shown in Fig. 4 over the

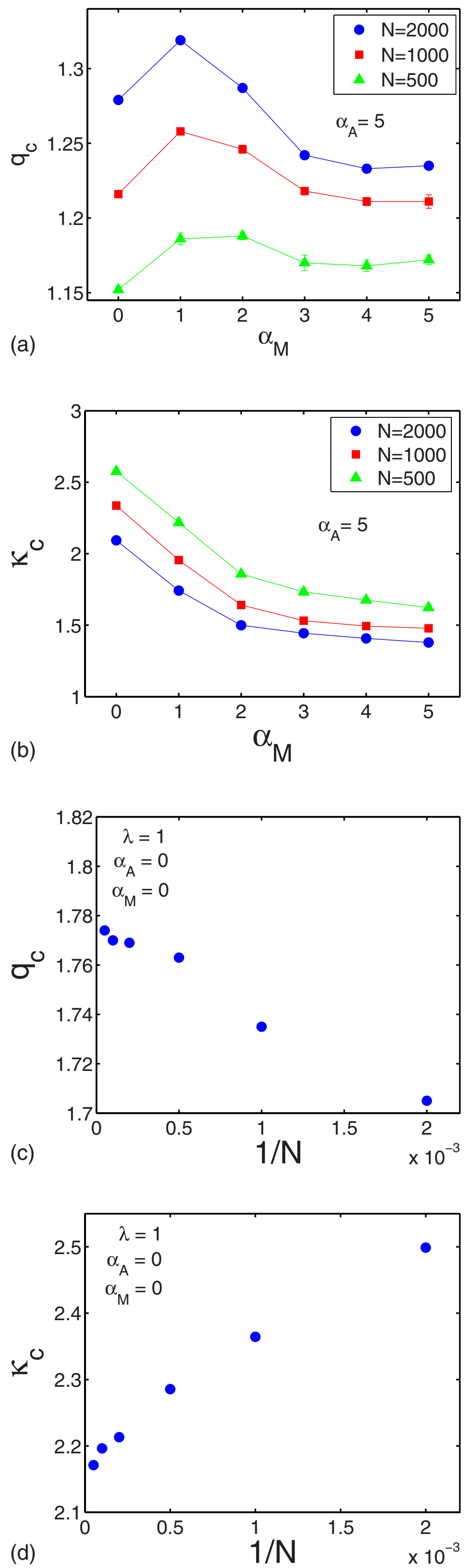

FIG. 5. (Color online) (a) q_c values of three system sizes for $\lambda=2$, $\alpha_A=5$, and α_M ranging from 0 to 5. (b) Same for κ_c. (c) and (d) show the same parameters as a function of network size N, for $\lambda=1$, $\alpha_A=\alpha_M=0$. For these parameters networks up to a size of $N=20\,000$ were possible.

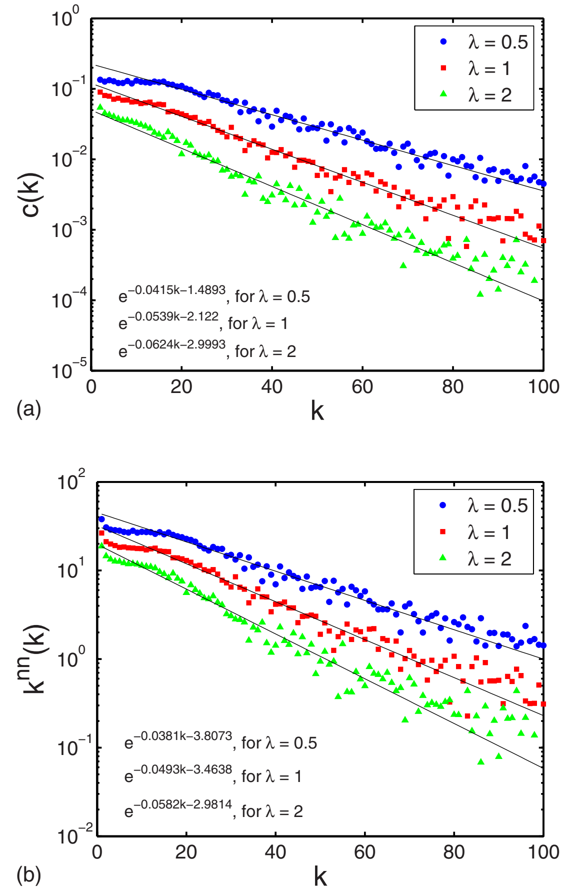

FIG. 6. (Color online) (a) Clustering coefficient $c(k)$ and (b) average nearest-neighbor degree k^{nn} for $\lambda=0.5,1,2$ and a fixed $\alpha_A=\alpha_M=0$ for $N=1000$. Averaging was done over 100 realizations.

parameter space. From top to bottom three values of λ are shown. The q_c index is declining in all three parameters, α_A, α_M, and λ. It eventually converges to a plateau in the α_A-α_M plane. The height of the plateau slowly decreases with higher λ, but remains above 1; $q_c=1$ corresponds to the exponential (ER) case. For low α_M there is a maximum of κ_c at about $\alpha_A\sim 3$; for larger α_M a plateau is forming for all α_A. This plateau remains constant as a function of λ. The quality of the q-exponential fit is demonstrated by the χ^2 test statistics per degree of freedom.

As in [4] we observe a finite size effect in the data. In Fig. 5(a) we show the dependence of the degree distribution parameters as a function of α_M for different system sizes for a fixed $\alpha_A=5$, and $\lambda=2$. The fits for κ_c are shown in Fig. 5(b).

We now turn to the clustering and neighbor connectivity of the emerging networks. In Fig. 6 we show the clustering coefficient c and the average neighbor connectivity k^{nn} as a function of k. For both quantities, the functional form of the decline with k is well fit with a two-parameter exponential fit, $\exp(-\epsilon_1\, k+\epsilon_2)$.

In Fig. 7 we show the fit parameters (a) ϵ_1 for $c(k)$ and (b) $k^{\mathrm{nn}}(k)$ for $\lambda=0.5$. For larger λ the clustering coefficients become drastically smaller, as expected for the $\lambda\to\infty$ and $\alpha_A\to 0$ limit. Fits for $\alpha_A>5$ and $\alpha_M>5$ become increasingly noisy and are omitted from the figure.

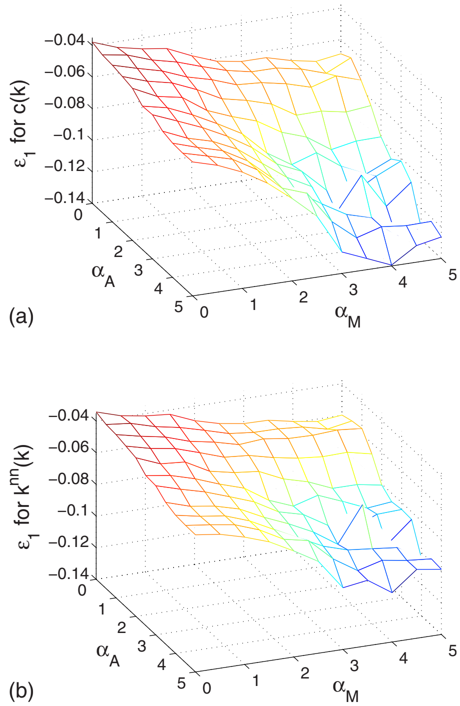

FIG. 7. (Color online) Exponential decay constants (a) ϵ_1 for $c(k)$ and (b) $k^{\mathrm{nn}}(k)$ over α_A and α_M for $\lambda=0.5$. The fit range was $k\in[1,100]$ and averages over 100 independent configurations have been taken. Fits for $\alpha_A>5$ and $\alpha_M>5$ become statistically insignificant.

In Fig. 8 we compare the global clustering coefficients from our model with those obtained from a random graph with the same dimensions (same number of nodes and links). For the Erdös-Renyi random graph the clustering coefficient is $C_{\mathrm{rand}}=\langle k\rangle/N-1$. The comparison makes clear that there is almost no attachment effects for $\alpha_A>3$ (i.e., negligible dependence from α_A), and a strong dependence on α_M and λ, as expected. A large number of real-world networks shows clustering coefficients which are within the range found in the presented model, i.e., $0<C<0.1$, which are all well above the corresponding ER values. For example, these include the WWW (C=0.1), power grid (C=0.08), protein interaction networks (C=0.02), the marine foodweb (C=0.08), or the Medline cooperation network (C=0.06), data from [10,29].

V. DISCUSSION

We have introduced a general network formation model which is able to recover, as particular instances, a large class of known network types. We checked that, to a very good approximation (high statistical significance), the resulting degree distributions exhibit q-exponential forms, with $q>1$. While a full theory of how complex networks are connected to $q\neq 1$ statistical mechanics is still missing, we provide further evidence that such a relation does indeed exist. As previously mentioned, if we associate a finite fixed energy or "cost" to every bond, and associate with each node one-half of the energy corresponding to its bonds (the other one-half corresponding to the other nodes linked by those same bonds), then the degree distribution can be seen as an energy distribution of the type emerging within nonextensive statistical mechanics (we recall that the stationary distribution obtained within Boltzmann-Gibbs statistics is of the exponential form, instead of the q-exponential one observed in scale-free networks). It might well be that the full understanding of this relation arises from the discrete nature of networks. The importance of appropriate values of $q\neq 1$ for systems living in topologies with a vanishing Lebesgue measure has been pointed out before [2]. This possibly makes phase space for certain nonextensive systems look like a network itself. In this view the basis of nonextensive systems could be related to a networklike structure of their phase space, explaining the ubiquity of q-exponential distribution functions in the world of networks.

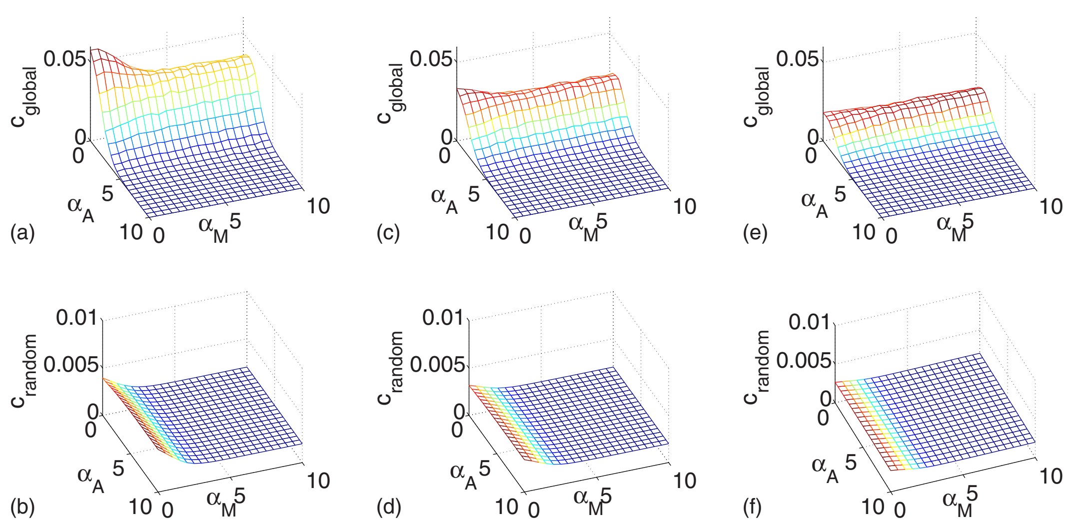

FIG. 8. (Color online) Global clustering coefficient in the α_A-α_M plane for $\lambda=0.5,1,2$ (from top to bottom) and $N=1000$ for the present model (a), (c), (e) and for an ER random graph with the same number of links and nodes (b), (d), (f). Averages were taken over 100 independent realizations.

Let us end by pointing out that, in variance with frequent such statements in the literature, the present model neatly illustrates that never-ending growth is *not* necessary for having networks that are (asymptotically) scale free. In this sense we are in full consensus with earlier work [17,21], q-exponential degree distributions do emerge for large enough networks which do *not* necessarily grow.

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